

Vacuum polarization in asymptotically anti-de Sitter black hole geometries

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We study the polarization of the vacuum for a scalar field, $\langle\phi^2\rangle$, on a asymptotically anti-de Sitter black hole geometry. The method we follow uses the WKB analytic expansion and point-splitting regularization, similarly to previous calculations in the asymptotically flat case. Following standard procedures, we write the Green function, regularize the initial divergent expression by point-splitting, renormalize it by subtracting geometrical counter-terms, and take the coincidence limit in the end. After explicitly demonstrating the cancellation of the divergences and the regularity of the Green function, we express the result as a sum of two parts. One is calculated analytically and the result expressed in terms of some generalized zeta-functions, which appear in the computation of functional determinants of Laplacians on Riemann spheres. We also describe some systematic methods to evaluate these functions numerically. Interestingly, the WKB approximation naturally organizes $\langle\phi^2\rangle$ as a series in such zeta-functions. We demonstrate this explicitly up to next-to-leading order in the WKB expansion. The other term represents the ‘remainder’ of the WKB approximation and depends on the difference between an exact (numerical) expression and its WKB counterpart. This has to be dealt with by means of numerical approximation. The general results are specialized to the case of Schwarzschild-anti-de Sitter black hole geometries. The method is efficient enough to solve the semi-classical Einstein’s equations taking into account the back-reaction from quantum fields on asymptotically anti-de Sitter black holes.

I. INTRODUCTION

Classically, matter influences gravity via its stress-energy tensor, $T_{\mu\nu}$, that appears as a source term in Einstein’s equations. It seems natural to expect that, if quantum fluctuations of matter fields and the curvature are sufficiently small, this picture may remain valid also in the semi-classical theory, where quantum fields propagate on a given classical geometry. In this case, the quantum field would couple to gravity via its stress-energy tensor, and its ‘back-reaction’ effects on the background spacetime would be described by the *semi-classical* Einstein’s equations:

$$G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle.$$

Even though the precise range of validity of the above equations cannot be quantified without knowledge of the full quantum gravity, one may reasonably expect that, as long as back-reaction effects are locally small, semi-classical Einstein’s equations may still provide an adequate description even if they lead to large cumulative effects, as in black hole evaporation.

In general, calculating semi-classical back-reaction effects is not an easy task, and it presents itself with a series of major difficulties. Ref. [1] lists three of them. The first is related to the fact that in curved spacetime no renormalization prescription can be uniquely given, and this generates an ambiguity in the expectation value of the

stress-energy tensor. Obviously, the semi-classical approximation cannot resolve this issue. The second problem is related to the fact that back-reaction effects introduce higher-derivative terms into the classical equations, possibly introducing spurious solutions. Therefore new criteria, beyond the semi-classical approximation, have to be invoked to select the physically relevant ones. A third problem stressed in [1] is that: ‘It is very difficult to compute $\langle T_{\mu\nu}\rangle$ ’. Even for highly symmetric spacetimes, or for particularly simple vacuum states, it is a formidable task to obtain $\langle T_{\mu\nu}\rangle$ even by numerical approximation.

The problem of quantum back-reaction is particularly interesting in the case of black holes, and, more generically, in spacetimes with horizons, because particle creation may lead to significant effects that may cause the evaporation of black holes. For this reason, since the original discovery of Hawking [2], the study of quantum effects from matter fields on black hole geometries has been the subject of much attention.

What is really necessary for studying the effect of the back reaction is the full averaged energy-momentum tensor. However, $\langle\phi^2\rangle$, which we will term for convenience *vacuum polarization*, already conveys much physical information, with the bonus of a slightly less cumbersome computation. Candelas was the first to calculate the polarization of the vacuum for massless, minimally coupled scalar fields on a Schwarzschild geometry outside the horizon, and obtained a renormalized expression for $\langle\phi^2\rangle$ using the point-splitting method [3]. The initial work of Ref. [3] has been extended in various directions. Further analysis of the renormalized vacuum polarization has been done in [4] and the expectation value of the

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energy-momentum tensor for the Hartle-Hawking state in the case of a conformally coupled scalar field on the same geometry has been obtained in [5, 6]. Other results were reported in [7], where the expression for the vacuum polarization has been extended inside the black hole horizon. In Ref. [8] Fawcett calculated numerically the energy-momentum tensor for conformal scalar fields and compared those results with Candelas' ones [3] and with Page's analytical approximation [9, 10]. Jensen and Ottewill have extended the above computation of the energy-momentum tensor to the electromagnetic field [11].

Since these initial works, the study of black hole radiance kept, and is still keeping, the interest in the problem alive. Novel results have been obtained and extended the original ones to different background geometries and higher spin fields. Anderson considered the case of a massive field with general coupling and calculated the vacuum polarization on a Schwarzschild background [12] and on an asymptotically-flat, static, spherically symmetric spacetime [13], both in the case of Hartle-Hawking vacuum. These results have then been extended and the full energy-momentum tensor has been evaluated in [14]. Based on the methods described in Ref. [12, 13, 14], Sushkov developed an analytical approximation and obtained $\langle\phi^2\rangle$ for Schwarzschild and wormholes spacetimes [15]. The analysis of quantum effects has also been extended to consider various aspects of quantum effects in rotating black hole geometries in [16, 17, 18, 19] and recently further work in the case of Reissner-Nordstrom [20] and lukewarm black holes has also been studied [21].

More recently, renewed interest in the quantum back reaction on black hole geometries has arisen in the context of the AdS/CFT correspondence. The five-dimensional brane world model proposed by Randall and Sundrum with an infinite warped extra-dimension [22] has been noticed to be possibly equivalent to the four-dimensional Einstein gravity with CFT corrections [23]. If we believe this correspondence, gravity in the brane world can be understood by inspecting the quantum back-reaction due to CFT. Applying the correspondence to black holes, a conjecture has been proposed: a brane localized black hole in Randall-Sundrum brane world model cannot stay static [24, 25]. In this context, the energy-momentum tensor due to a free scalar field on a four-dimensional Schwarzschild background in the Boulware state, which is compatible with the asymptotic flat condition, was computed, and it was confirmed that the quantum back-reaction diverges on the horizon [26]. Although there have been many attempts to examine whether this conjecture is correct or not [27, 28, 29, 30, 31, 32, 33, 34], we have not reached any definite conclusion yet. The original conjecture is about asymptotically flat brane geometries. One of the present authors suggested that one may be able to shed new light on this issue by extending the setup to asymptotically AdS brane geometries [35, 36]. What will happen on the CFT side has already been discussed by Haw-

king and Page many years ago [37]. In this case it is expected that a static black hole solution exists even after taking into account the quantum back-reaction if the size of the black hole is sufficiently large. However, such a self-consistent quantum black hole solution has not been realized numerically so far.

The above reasons provided enough motivation for us to consider the effects of quantum fields on asymptotically anti-de Sitter (AdS) black hole geometries. We will not solve the full back-reaction problem in this notes, rather we will focus on the computation of the vacuum polarization for massive, non-minimally coupled scalar fields, leaving the details of the full energy-momentum tensor calculation and of the effects of the quantum back-reaction to a forthcoming publication [38].

The strategy presented in this paper is valid for a general spherically symmetric, asymptotically AdS spacetime, but, for brevity, we focus our attention to the case of Schwarzschild-anti-de Sitter (Sch-AdS) black holes, described by the following form line element:

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \quad (1)$$

where

$$f(r) = 1 - \frac{1}{m_p^2} \frac{2M}{r} + r^2 k^2, \quad (2)$$

$m_p = G^{-1/2}$ is the Planck mass in natural units and k^{-1} is the AdS curvature length. The horizon is located at $r = r_h$, where $f(r_h) = 0$. Customarily, the time coordinate is complexified, $\tau = it$, rendering the metric positive definite outside of the horizon. The apparent singularity at the horizon is removed by regarding the τ coordinate as periodic with period β :

$$\beta = 4\pi \left[\frac{df}{dr} \right]_{r=r_h}^{-1},$$

that, for Sch-AdS black holes, becomes

$$\beta = 4\pi \frac{r_h}{1 + 3k^2 r_h^2}.$$

Analogously to the asymptotically flat cases, asymptotically AdS black holes have a characteristic temperature, $T = \beta^{-1}$, and an intrinsic entropy equal to a quarter of the area of the event horizon. However, unlike the asymptotically flat black holes, there exists a minimum temperature which occurs when its size is of the order of the characteristic radius of the AdS space. As a working example, we will consider the above AdS black holes and the numerical analysis will be specialized to this case.

The results we will present are technical and mainly follow the direct procedure outlined by Candelas in the case of Schwarzschild black holes. The Green function is expressed as a sum of the products of the normal modes, therefore the first step in the computation requires the homogeneous solutions of the matter field equations to obtain a complete set of solutions. The choice of the

normal modes selects a specific vacuum state. Although the wave equation is separable in the spherically symmetric case, it is not in general analytically soluble, and some numerical analysis is necessary. However, to practically perform renormalization of the Green function, which originally diverges quartically, we need some analytic approximation to the normal modes. For this purpose, the WKB approximation is useful. In the following we illustrate how to obtain the WKB approximants for the normal modes, highlighting the differences with the asymptotically flat case.

After this initial part, we discuss the renormalization of the Green function. To subtract the divergent geometrical counter-terms, we first need to regularize the divergent sum. For this purpose, an appropriate procedure is the point splitting regularization, developed in general by Christensen [39]. Relaxing the regularization after subtracting the divergent pieces, we obtain a renormalized expression for the coincidence limit of the Green function, which is formally divided into two parts: one amenable to analytical computation and one that requires numerical evaluation. The latter part can be arranged so as to get fast convergence in the mode sum.

The point splitting method guarantees the regularity of the Green function. Customarily the explicit proof of the regularity is not illustrated. Despite of the apparent superfluity, we explicitly perform this computation, which, aside of being a non-trivial check of a complex algebraic calculation, will turn out to be useful to convince the reader of our systematic way of evaluating the renormalized vacuum polarization. In fact, the renormalized propagator is made of various divergent parts, which, combined with each other, lead to a regular expression. Recognizing the structure of the divergences for each of these pieces allows us to perform a piece-by-piece renormalization. This will allow us to handle the numerical computation in a more convenient way. In fact, as we will see, the analytical part of the propagator can be expressed in terms of certain regularized ζ -functions, which appear in the computation of determinants of Laplacians on Riemann spheres [40]. The conclusive computation of the vacuum polarization is illustrated in Sec. V where the numerical and analytical calculations are presented in detail. Our final remarks close the paper.

II. GREEN FUNCTION

The Euclidean Green function for a scalar field satisfies the following equation:

$$(\square - m^2 - \xi R) G_E(X, X') = -\sqrt{g}\delta(X, X') .$$

Choosing as vacuum state the Hartle-Hawking one, it can be expressed as

$$G_E(X, X') = \frac{1}{\beta} \sum_n e^{2\pi i n(\tau - \tau')/\beta}$$

$$\times \sum_l \frac{(2l+1)}{4\pi} P_l(\cos \gamma) G_{nl}(r, r') , \quad (3)$$

which is characterized by the regularity and the periodicity along the τ direction with period β . Here, $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$. Writing the delta function as

$$\delta(\tau, \tau') = \frac{1}{\beta} \sum_n e^{2\pi i n(\tau - \tau')/\beta} ,$$

one can find that the function G_{nl} satisfies

$$\left\{ \frac{d}{dx} \left(x^2 f \frac{d}{dx} \right) - l(l+1) - \frac{x^2 \omega_n^2}{f} - (\hat{m}^2 + \xi \hat{R}) x^2 \right\} G_{nl}(x, x') = -\delta(x - x') . \quad (4)$$

In the above equation we have rescaled the radial coordinate, the mass and the curvature as:

$$x = kr , \quad \hat{m} = m/k , \quad \hat{R} = k^{-2} R ,$$

and defined

$$\omega_n = \alpha n , \quad (5)$$

with

$$\alpha = \frac{2\pi}{k\beta} ,$$

which for Sch-AdS black holes takes the following form:

$$\alpha = \frac{1}{2} \left(\frac{1}{x_h} + 3x_h \right) ,$$

where $x_h = kr_h$. The function $G_{nl}(x, x')$ can be written in the usual way as the product of the two independent solutions of the homogeneous equation associated with Eq. (4):

$$\left\{ \frac{d}{dx} \left(x^2 f \frac{d}{dx} \right) - l(l+1) - \frac{x^2 \omega_n^2}{f} - (\hat{m}^2 + \xi \hat{R}) x^2 \right\} \varphi(x) = 0 . \quad (6)$$

Using tortoise coordinates,

$$dx_* = dx/f ,$$

the homogeneous equation can be written as

$$\left[\frac{d^2}{dx_*^2} - \omega_n^2 - \left(\frac{l(l+1)}{x^2} + \frac{f'}{x} + \hat{m}^2 + \xi \hat{R} \right) f \right] (x\varphi) = 0 , \quad (7)$$

where $'$ represents derivative with respect to x . It is instructive to examine the behavior of the solutions near the horizon and at infinity. In the near horizon region the solutions are determined by

$$\left[\frac{d^2}{dx_*^2} - \omega_n^2 \right] (x\varphi) \sim 0 . \quad (8)$$

This leads, as in the asymptotically flat case, to exponential solutions,

$$\varphi \sim e^{\pm \omega_n x_*} / x, \quad (9)$$

one of them being regular (at the horizon). At infinity (large- x) one finds

$$\left[\frac{d^2}{dx^2} + \frac{4}{x} \frac{d}{dx} - \frac{1}{x^2} (\hat{m}^2 + \xi \hat{R}) \right] \varphi \sim 0, \quad (10)$$

which, differently from the asymptotically flat case, admits solutions of the form

$$\varphi \sim x^{-\frac{3}{2} \pm \frac{1}{2} \sqrt{9+4(\hat{m}^2 + \xi \hat{R})}}. \quad (11)$$

Going back to the original equation (4), its solutions should be chosen by specifying their asymptotic behavior so that they are regular at infinity and on the horizon. We indicate as $p_l^n(r)$ the solution regular on the horizon and as $q_l^n(r)$ the one regular at infinity. The WKB approximation of these solutions will be discussed in detail in the next section. Notice that, in the Schwarzschild case, Eq. (4) takes the form of Heun equation, with two regular singular points and one irregular singularity at infinity. The 4D Sch-AdS case is different. Eq. (4) has five regular singular points, with infinity being a regular singularity.

The $n = 0$, conformally coupled case is somewhat special, because at the horizon the zero order WKB approximant vanishes in the limit $\xi = 1/6$ and $m = 0$. In the $n = 0$ case, the behavior at the horizon can be easily understood by re-expressing the mode equation in terms of the logarithmic derivative of φ near the horizon. It takes the form

$$x^2 f' \frac{d \ln \varphi}{dx} - l(l+1) - 2x^2 = 0,$$

from which it is easy to read off the behavior at $x = x_h$:

$$\left[\frac{\varphi'}{\varphi} \right]_{x_h} = \frac{l(l+1) - 2x_h^2}{x_h + 3x_h^3}.$$

Standard procedure leads to the following expression for the Green function:

$$G_E(X, X') = \frac{\alpha k^2}{2\pi} \sum_n e^{in\alpha\epsilon} \sum_l \frac{(2l+1)}{4\pi} P_l(\cos \gamma) \times \frac{p_l^n(x_<) q_l^n(x_>)}{x^2 f (q_l^n dp_l^n/dx - p_l^n(x) dq_l^n/dx)},$$

where $\epsilon = k(\tau - \tau')$. The above expression is yet formal, because we have not found any explicit solution for the mode functions. In fact, as one can easily guess, such a solution can only be found numerically or by using some approximation method.

III. WKB SOLUTIONS

A convenient way to obtain an analytic expression for the solution of Eq. (6) is to use the WKB approximation. As we will see, this approximation is suitable and sufficient to perform all the renormalization procedure that we will need later.

The WKB method can be implemented by writing the solution as

$$\varphi(x) = x^{-3/2} W^{-\eta} e^{\pm \int_{x_h}^x W(x') h(x') dx'}, \quad (12)$$

with $\eta > 0$. The $+$ sign refers to the solution regular at the horizon, which we have called $p_l^n(x)$ and the $-$ sign to $q_l^n(x)$. The overall factor $x^{-3/2}$ is multiplied so that the asymptotic form at $x \rightarrow \infty$ becomes compatible with the WKB ansatz: $\varphi \sim x^{-\frac{3}{2}} e^{\pm \frac{1}{2} \sqrt{9+4(\hat{m}^2 + \xi \hat{R})} \log x}$. Substituting the previous ansatz in the homogeneous equation associated with (4), one easily finds

$$\begin{aligned} 0 = & W^2 \pm \left(-\frac{3}{xh} + \frac{h'}{h^2} + \frac{(x^2 f)'}{x^2 f h} \right) W \pm \frac{1-2\eta}{h} W' \\ & + \eta \left(\frac{3}{xh^2} - \frac{(x^2 f)'}{x^2 f h^2} \right) \frac{W'}{W} - \frac{\eta}{h^2} \frac{W''}{W} \\ & + \frac{\eta(1+\eta)}{h^2} \frac{W'^2}{W^2} + \frac{15}{4x^2 h^2} - \frac{3(x^2 f)'}{2x^3 f h^2} - \frac{l(l+1)}{x^2 f h^2} \\ & - \frac{\omega_n^2}{f^2 h^2} + \hat{m}^2 \frac{f}{x^2} + \xi \hat{R} \frac{f}{x^2}. \end{aligned}$$

We choose the coefficient η and the function h so as to cancel the term with (\pm) -signature. Hence, we have $\eta = 1/2$ and $h = x/f$, and the WKB equation simplifies to

$$W^2 = \varpi + \sigma + a_1 \frac{W'}{W} + a_2 \frac{W'^2}{W^2} + a_3 \frac{W''}{W}, \quad (13)$$

where

$$\begin{aligned} \varpi &\equiv \left[\left(l + \frac{1}{2} \right)^2 - \frac{1}{4} \right] \frac{f}{x^4} + \frac{\omega_n^2}{x^2}, \\ \sigma &\equiv \frac{3}{2} \frac{f f'}{x^3} - \frac{3}{4} \frac{f^2}{x^4} + \hat{m}^2 \frac{f}{x^2} + \xi \hat{R} \frac{f}{x^2}, \end{aligned}$$

and we have defined

$$a_1 \equiv \frac{f}{2x^2} \left(f' - \frac{f}{x} \right), \quad a_2 \equiv -\frac{3}{4} \frac{f^2}{x^2}, \quad a_3 \equiv \frac{f^2}{2x^2}.$$

As it is well-known, the WKB method expresses the solution iteratively as

$$W = W^{(0)} + W^{(1)} + W^{(2)} + \dots$$

The leading order term is

$$W^{(0)} = \sqrt{\Phi(l)} \equiv \sqrt{\varpi + \sigma},$$

the next-to-leading order correction is computed by adding the derivative terms in (13) evaluated for $W =$

$W^{(0)}$, and so on, iteratively, to the desired order. In the following, everything is calculated up to next-to-leading order, *i.e.* $W^{(1)}$.

To be more precise, we introduce the following truncated WKB approximation of $1/W$:

$$\frac{1}{W_n(z)} \equiv \frac{1}{\Phi(z)^{1/2}} - \frac{\Psi(z)}{4\Phi(z)^{3/2}}, \quad (14)$$

where the function $\Psi(z)$ conveys the next-to-leading order WKB corrections and is defined by

$$\Psi \equiv a_1 \frac{\Phi'}{\Phi} + \tilde{a}_2 \frac{\Phi'^2}{\Phi^2} + a_3 \frac{\Phi''}{\Phi}, \quad (15)$$

where

$$\tilde{a}_2 \equiv \frac{a_2 - a_3}{2} = -\frac{5}{8} \frac{f^2}{x^2}. \quad (16)$$

Going to higher order in the WKB expansion means adding higher order terms obtained by reiterating the above procedure. The explicit WKB expansions are lengthy and we will not report them here, but they can easily be handled by any symbolic algebra manipulation program.

One can easily check the quality of the WKB approximation by comparing it with the numerical solutions. It turns out that the numerical solutions are well reproduced by the WKB approximation already for small values of n and l , as it is easy to argue. In fact, expanding W to next-to-leading order, one can easily see that the remainder in $1/W$ is of order $O(l^{-5}, n^{-5})$.

IV. COINCIDENCE LIMIT

Our main goal is to compute the coincidence limit of the Green function which will provide us the expression for $\langle \phi^2 \rangle$. Here we follow the same procedure that has been used in the Schwarzschild case. First take the partial coincidence limit, by setting $r = r'$ and $\Omega = \Omega'$. In this case the expression for the Green function simplifies to

$$G_E(X, X') = \frac{k^2 \alpha}{8\pi^2} \sum_{n=0}^{\infty} e^{in\alpha\epsilon} \sum_{l=0}^{\infty} \frac{(l+1/2)}{x^3 \tilde{W}_n(l)}, \quad (17)$$

where

$$\frac{1}{\tilde{W}_n(l)} \equiv \frac{2x}{f} \left(\frac{d \ln q_l^n(x)}{dx} - \frac{d \ln p_l^n(x)}{dx} \right)^{-1}.$$

In the above expression the coincidence limit cannot be taken in a straightforward way because the sums over l and n are divergent. The divergence related to the l summation is not a serious one and can be by-passed easily by noticing that the Green function is a finite and well-defined object as long as $X \neq X'$, *i.e.* $\epsilon \neq 0$. The trick commonly used is to add a multiple of $\delta(\tau - \tau')$ (and its derivatives if necessary) which, as long as $\tau \neq \tau'$, does not alter the result. Basically, we can add terms of the form:

$$\frac{k^2 \alpha}{8\pi^2} \sum_n e^{in\alpha\epsilon} \sum_l R_l(x)$$

where a function $R_l(x)$ independent of n can be chosen freely. By looking at the asymptotic behavior of the solution for large l , which can be obtained from the WKB approximation,

$$\tilde{W}_n(l) \sim W_n(l) \sim (l+1/2)f^{1/2}x^{-2} + O((l+1/2)^{-1}),$$

we can easily find that choosing $R_l(x) = 1/(x\sqrt{f})$ is sufficient to remove the divergence in the summation over l , leading to

$$G_E(X, X') = \frac{k^2 \alpha}{8\pi^2} \sum_{n=0}^{\infty} e^{in\alpha\epsilon} \sum_{l=0}^{\infty} \left[\frac{l+1/2}{x^3 \tilde{W}_n(l)} - \frac{1}{x\sqrt{f}} \right]. \quad (18)$$

The above expression is still (ultraviolet) divergent in the coincidence limit due to the summation over n . In a general spherically symmetric space-time, the expression that needs to be subtracted from the Green function, before taking the $\epsilon \rightarrow 0$ limit, is known to be given by [13, 39]

$$G_{div} = \frac{k^2}{16\pi^2} \left\{ \frac{4}{\epsilon^2 f} + \left(\hat{m}^2 + \left[\xi - \frac{1}{6} \right] \hat{R} \right) \left(\ln \left[\frac{\hat{m}^2 f \epsilon^2}{4} \right] + 2\gamma_E \right) - \hat{m}^2 + \frac{1}{12f} \left(\frac{df}{dr} \right)^2 - \frac{1}{6} \frac{d^2 f}{dr^2} - \frac{1}{3r} \frac{df}{dr} \right\}, \quad (19)$$

where γ_E is the Euler's constant. As shown in Ref. [13], a more suitable way to combine the above counter-term with the original divergent expression for the Green function is to re-express (19), by using the Abel-Plana summation formula, as

$$G_{div} = \frac{k^2 \alpha}{8\pi^2} \left[- \sum_{n=1}^{\infty} e^{in\alpha\epsilon} \left\{ \frac{2\omega_n}{f} + \left[\hat{m}^2 + \left(\xi - \frac{1}{6} \right) \hat{R} \right] \frac{1}{\omega_n} \right\} + \Delta_1 + \Delta_2 \right], \quad (20)$$

where

$$\Delta_1 \equiv - \sum_{n=1}^{\infty} \left\{ \frac{2}{f} \left(\sqrt{\omega_n^2 + \hat{m}^2 f} - \omega_n - \frac{\hat{m}^2 f}{2\omega_n} \right) - \left(\xi - \frac{1}{6} \right) \hat{R} \left(\frac{1}{\sqrt{\omega_n^2 + \hat{m}^2 f}} - \frac{1}{\omega_n} \right) \right\} , \quad (21)$$

and $\Delta_2 = \Delta_{2,1} + \Delta_{2,2} + \Delta_{2,3}$ with

$$\begin{aligned} \Delta_{2,1} &\equiv \frac{\hat{m}^2}{2\alpha} \ln(\hat{m}^2 f) - \frac{\hat{m}^2}{\alpha} \ln\left(\alpha + \sqrt{\alpha^2 + \hat{m}^2 f}\right) + \frac{\hat{m}^2}{16\pi^2} \\ &\quad + \frac{2i}{f} \int_0^\infty \frac{dt}{e^{2\pi t} - 1} \left(\sqrt{\alpha^2(1+it)^2 + \hat{m}^2 f} - \sqrt{\alpha^2(1-it)^2 + \hat{m}^2 f} \right) , \\ \Delta_{2,2} &\equiv \frac{(\xi - 1/6)\hat{R}}{2} \left[\frac{1}{\sqrt{\alpha^2 + \hat{m}^2 f}} - \frac{2}{\alpha} \ln\left(\alpha + \sqrt{\alpha^2 + \hat{m}^2 f}\right) + \frac{1}{\alpha} \ln(\hat{m}^2 f) \right. \\ &\quad \left. + 2i \int_0^\infty \frac{dt}{e^{2\pi t} - 1} \left(\frac{1}{\sqrt{\alpha^2(1+it)^2 + \hat{m}^2 f}} - \frac{1}{\sqrt{\alpha^2(1-it)^2 + \hat{m}^2 f}} \right) \right] , \\ \Delta_{2,3} &\equiv \frac{1}{2\alpha} \left(\frac{1}{12f} f'^2 - \frac{1}{6} f'' - \frac{1}{3r} f' - \hat{m}^2 \right) . \end{aligned}$$

After subtracting (20) from (18), we obtain an expression for the renormalized Green function, and its full coincidence limit $\epsilon \rightarrow 0$, which provides the renormalized vacuum polarization, can be taken:

$$\begin{aligned} \langle \phi^2(X) \rangle = G_E^{(ren)}(X, X) &= \frac{k^2 \alpha}{8\pi^2} \left\{ -\Delta_1 - \Delta_2 + \sum_{l=0}^{\infty} \left((l+1/2) \frac{1}{x^3 \tilde{W}_0(l)} - \frac{1}{x\sqrt{f}} \right) \right. \\ &\quad \left. + 2 \sum_{n=1}^{\infty} \left[\sum_{l=0}^{\infty} \left((l+1/2) \frac{1}{x^3 \tilde{W}_n(l)} - \frac{1}{x\sqrt{f}} \right) + \frac{2}{f} \omega_n + \left(\hat{m}^2 + (\xi - 1/6)\hat{R} \right) \frac{1}{\omega_n} \right] \right\} . \quad (22) \end{aligned}$$

In the above relation we have separated, for convenience, the contribution from $n = 0$. As shown in [3], this $n = 0$ contribution vanishes in the Schwarzschild case, but it does not in the asymptotically AdS case.

V. VACUUM POLARIZATION

A. Summation over l and Regularity of the Green function

We explain how we perform the summation over l in (22) in this subsection. At the same time, we explicitly demonstrate the finiteness of the summation over n . Expression (22) should be, by construction, finite. However, as one can immediately notice, the contribution of each term in Eq. (22) is not separately finite. We need to combine them before taking the summation so as to cancel with each other. In the following we will demonstrate the finiteness of (22) explicitly. This, in principle, is not necessary, since the point-splitting regularization with subtraction of the appropriate counter-terms guarantees the finiteness of the renormalized Green function. However, aside from being a non-trivial check of the calculations, knowing explicitly how the various divergent pieces cancel with each other suggests a convenient strategy for the subsequent computation.

First of all, let us rearrange the renormalized Green function by adding and subtracting its WKB counterpart:

$$\langle \phi^2(X) \rangle = G_E^{(ren)}(X, X) = \frac{k^2 \alpha}{8\pi^2} \{ -\Delta_1 - \Delta_2 + \Upsilon_0 + \Sigma_1 + \Sigma_2 \} , \quad (23)$$

where

$$\begin{aligned} \Sigma_1 &\equiv \sum_{l=0}^{\infty} (l+1/2) \left(\frac{1}{x^3 \tilde{W}_0(l)} - \frac{1}{x^3 W_0(l)} \right) + 2 \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} (l+1/2) \left(\frac{1}{x^3 \tilde{W}_n(l)} - \frac{1}{x^3 W_n(l)} \right) , \\ \Sigma_2 &\equiv 2 \sum_{n=1}^{\infty} \left[\Upsilon_n + \frac{2}{f} \omega_n + \left(\hat{m}^2 + \left(\xi - \frac{1}{6} \right) \hat{R} \right) \frac{1}{\omega_n} \right] , \end{aligned} \quad (24)$$

and

$$\Upsilon_n \equiv \sum_{l=0}^{\infty} \left(\frac{l+1/2}{x^3 W_n(l)} - \frac{1}{x\sqrt{f}} \right). \quad (25)$$

The terms other than Σ_2 are manifestly finite. The term Σ_2 is cumbersome to evaluate, although straightforward. Once again, we can proceed in the standard way. Making use of the Abel-Plana summation formula, we rewrite Υ_n as

$$\Upsilon_n = \frac{1}{x^3} \left[\frac{1}{4 W_n(0)} + \left(\int_0^{\infty} \left(\frac{z+1/2}{W_n(z)} - \frac{x^2}{\sqrt{f}} \right) dz - \frac{x^2}{2\sqrt{f}} \right) + i \int_0^{\infty} \frac{dz}{e^{2\pi z} - 1} \left(\frac{iz+1/2}{W_n(iz)} - \frac{-iz+1/2}{W_n(-iz)} \right) \right]. \quad (26)$$

The contribution to Σ_2 from the first term in the square brackets of Eq. (26) is

$$\begin{aligned} \mathcal{P}_1 &\equiv \frac{1}{2x^3} \sum_{n=1}^{\infty} \frac{1}{W_n(0)} \\ &= \frac{1}{2x^3} \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{\Phi(0)}} - \frac{\Psi(0)}{4\Phi^{3/2}(0)} \right) \\ &= \frac{1}{4x^2} \left\{ 2\mathcal{Z}_1 + (xa_1 - 2\tilde{a}_2 - 3a_3) \mathcal{Z}_3 \right. \\ &\quad \left. - x^2 \left[\left(\frac{xa_1}{2} - 2\tilde{a}_2 \right) (2\sigma + x\sigma') \right. \right. \\ &\quad \left. \left. - a_3 \left(3\sigma - \frac{x^2\sigma''}{2} \right) \right] \mathcal{Z}_5 \right. \\ &\quad \left. - \frac{x^4\tilde{a}_2}{2} (2\sigma + x\sigma')^2 \mathcal{Z}_7 \right\}, \end{aligned} \quad (27)$$

where we have substituted $1/W_n(z)$, given by Eq. (14), in the first equality and the explicit expression of Ψ , given by Eq. (15), in the second equality. Here we have introduced generalized ζ -functions of Epstein-Hurwitz type, defined by

$$\mathcal{Z}_q \equiv \sum_{n=1}^{\infty} (\omega_n^2 + x^2\sigma(x))^{-q/2}, \quad (28)$$

which occur in the computation of functional determinants of Laplacians on Riemann spheres (Some explicit applications can be found, for instance, in Refs. [40, 41].)

We refer to the n -th term of the divergent piece as $\text{div}[\mathcal{P}]_n$. To be precise, here we define $\text{div}[\mathcal{P}]_n$ by the two leading terms in the large n expansion in the form of $b_1\omega_n + b_2\omega_n^{-1}$, where b_1 and b_2 are constants. In the above expression (27) only the term with \mathcal{Z}_1 is divergent. Hence, we can easily extract the divergent part of \mathcal{P}_1 as

$$\text{div}[\mathcal{P}_1]_n = \frac{1}{2x^2\omega_n}. \quad (29)$$

All the sub-leading terms are finite.

The next term gives

$$\mathcal{P}_2 \equiv \frac{2}{x^3} \sum_{n=1}^{\infty} \left[\int_0^{\infty} \left(\frac{z+1/2}{W_n(z)} - \frac{x^2}{\sqrt{f}} \right) dz - \frac{x^2}{2\sqrt{f}} \right]$$

$$= \mathcal{P}_{2,1} + \mathcal{P}_{2,2}, \quad (30)$$

where we have defined

$$\begin{aligned} \mathcal{P}_{2,1} &\equiv \frac{2}{x^3} \sum_{n=1}^{\infty} \left[\int_0^{\infty} \left(\frac{z+1/2}{\Phi^{1/2}} - \frac{x^2}{\sqrt{f}} \right) dz - \frac{x^2}{2\sqrt{f}} \right], \\ \mathcal{P}_{2,2} &\equiv -\frac{1}{x^3} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{(z+1/2)\Psi}{\Phi^{3/2}} dz. \end{aligned}$$

Here the integrations over z can be performed easily. After integration over z , $\mathcal{P}_{2,1}$ reduces to

$$\mathcal{P}_{2,1} = -\frac{2}{f} \mathcal{Z}_{-1}. \quad (31)$$

Hence, we can extract the divergent part of $\mathcal{P}_{2,1}$ as

$$\text{div}[\mathcal{P}_{2,1}]_n = -\frac{2}{f} \left(\omega_n - \frac{\sigma}{2\omega_n} \right). \quad (32)$$

Also, we perform the integration over z in $\mathcal{P}_{2,2}$, substituting the explicit expression of Ψ , to obtain

$$\begin{aligned} \mathcal{P}_{2,2} &= \frac{a_1}{6f} \left\{ x \left(10 - 2\frac{xf'}{f} \right) \mathcal{Z}_1 \right. \\ &\quad \left. - x^3 (2\sigma + x\sigma') \mathcal{Z}_3 \right\} \\ &\quad - \frac{\tilde{a}_2}{30f} \left\{ 4 \left(43 - \frac{18xf'}{f} + 2\frac{x^2f'^2}{f^2} \right) \mathcal{Z}_1 \right. \\ &\quad \left. - 4x^2 \left(7 - \frac{xf'}{f} \right) (2\sigma + x\sigma') \mathcal{Z}_3 \right. \\ &\quad \left. + 3x^4 (2\sigma + x\sigma')^2 \mathcal{Z}_5 \right\} \\ &\quad - \frac{a_3}{6f} \left\{ \left(46 - 16\frac{xf'}{f} + 2\frac{x^2f''}{f} \right) \mathcal{Z}_1 \right. \\ &\quad \left. - (6x^2\sigma - x^4\sigma'') \mathcal{Z}_3 \right\}. \end{aligned} \quad (33)$$

Since the divergence in the above expression is contained only in the terms proportional to \mathcal{Z}_1 , we can easily extract the divergent part as

$$\text{div}[\mathcal{P}_{2,2}]_n = \frac{1}{6\omega_n} \left\{ -\frac{13f}{2x^2} + \frac{5f'}{x} - f'' \right\}, \quad (34)$$

where we have substituted explicit forms of a 's given in Eqs. (14) and (16).

The last term to evaluate is

$$\mathcal{P}_3 = \frac{2i}{x^3} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dz}{e^{2\pi z} - 1} \left(\frac{iz + 1/2}{W_n(iz)} - \frac{-iz + 1/2}{W_n(-iz)} \right). \quad (35)$$

Due to the exponential fall-off, the dominant contribution to the integral comes from the $z \sim 0$ region. Hence, we can evaluate this term by expanding the part enclosed by the parentheses as

$$\left(\frac{iz + 1/2}{W_n(iz)} - \frac{-iz + 1/2}{W_n(-iz)} \right) = -i \sum_{j=1}^{\infty} c_{nj} z^{2j-1}. \quad (36)$$

The convergence of this series will be fast for large n , while it will be slow for small n . Therefore we divide \mathcal{P}_3 into two parts as

$$\mathcal{P}_3 = \mathcal{P}_{3,1} + \mathcal{P}_{3,2}, \quad (37)$$

where

$$\begin{aligned} \mathcal{P}_{3,1} &\equiv \frac{2}{x^3} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dz}{e^{2\pi z} - 1} \sum_{j=1}^{j_{\max}} c_{nj} z^{2j-1}, \\ \mathcal{P}_{3,2} &\equiv \frac{2i}{x^3} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{dz}{e^{2\pi z} - 1} \left\{ \left(\frac{iz + 1/2}{W_n(iz)} - \frac{-iz + 1/2}{W_n(-iz)} \right) + i \sum_{j=1}^{j_{\max}} c_{nj} z^{2j-1} \right\}, \end{aligned} \quad (38)$$

with an appropriately chosen value of j_{\max} . Then, on one hand, $\mathcal{P}_{3,2}$ can be evaluated numerically. As long as j_{\max} is sufficiently large, the summation over n converges rapidly. On the other hand, we can perform the integration over z in $\mathcal{P}_{3,1}$ analytically term by term using the formula $\int_0^{\infty} z^{2j-1} (e^{2\pi z} - 1)^{-1} dz = \Gamma(2j) \zeta_R(2j) / (2\pi)^{2j} = (-1)^{j-1} B_{2j} / 4j$, where ζ_R is the Riemann ζ -function and $B_1 = 1/2, B_2 = 1/6, B_3 = 0, B_4 = 1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = 1/30, \dots$ are the Bernoulli numbers. Finally, we obtain

$$\mathcal{P}_{3,1} \equiv \frac{2}{x^3} \sum_{n=1}^{\infty} \sum_{j=1}^{j_{\max}} c_{nj} \frac{(-1)^{j-1}}{4j} B_{2j}. \quad (39)$$

Since c_{nj} is $O(1/\omega_n^{2j-1})$ for large n , only the part with $j = 1$ is divergent. Thus, using $c_{n1} \simeq -2/W_n(0)$ and $B_2 = 1/6$, we have

$$\text{div} [\mathcal{P}_3]_n = -\frac{1}{6x^2 \omega_n}. \quad (40)$$

It is now a matter of trivial algebra to combine together (29), (32), (34) and (40), to show that they cancel the contribution from the last line in (22), leaving a well-behaved expression.

The same procedure applies in computing Υ_0 , too. We just need to eliminate the summation over n setting n to

zero in the expressions for \mathcal{P} 's. For the evaluation of the counter part of \mathcal{P}_3 in computing Υ_0 , we do not need to divide it into two pieces like $\mathcal{P}_{3,1}$ and $\mathcal{P}_{3,2}$ since there is no infinite summation over n .

The remaining task is composed of two parts; numerical calculation and analytic summation over n . The former is necessary for $\Delta_2, \Upsilon_0, \mathcal{P}_{3,2}$, but infinite summation over n is unnecessary for these terms. Only the leading finite number of terms give sufficiently precise approximation. The other terms, i.e. Δ_1 and the other \mathcal{P} 's, can be handled completely analytically. Moreover, from the above demonstration, we find that all the summations over n which appear in \mathcal{P} 's are written in terms of \mathcal{Z}_n and its counter-terms in the form $\sum_n b_1 \omega_n + b_2 \omega_n^{-1}$. \mathcal{Z}_{-1} and \mathcal{Z}_1 contain divergences, and hence they require the counter-terms. The coefficients of the counter-terms, b_1 and b_2 , are appropriately chosen so as to cancel these divergences. Hence, we can simply define a regularized \mathcal{Z}_{-1} and \mathcal{Z}_1 by including the counter-terms proportional to $\sum_n \omega_n$ or $\sum_n \omega_n^{-1}$ so that they are finite. We denote such regularized \mathcal{Z}_{-1} and \mathcal{Z}_1 as $\tilde{\mathcal{Z}}_{-1}$ and $\tilde{\mathcal{Z}}_1$, respectively. Then, summing up the regularized pieces gives the correctly renormalized value.

B. Summation over n

In the previous subsections, we performed the summation over l in the renormalized expression for $\langle \phi^2 \rangle$, Eq. (23). The expression is composed of two parts; one is the part that requires numerical evaluation and the other is the part that requires infinite summation over n . In this subsection we will carry out the evaluation of these expressions.

It is straightforward to perform the necessary numerical computations. The term Σ_1 requires evaluating \tilde{W}_n and hence the exact mode functions, $p_l^n(x)$ and $q_l^n(x)$. For this term, summations over l and n are also necessary. In the practical numerical computation we need to truncate these summations at finite l and n . However, this term consists of the difference between \tilde{W}_n and its WKB approximant W_n , and hence it is the remainder of the approximation in this sense. Since the WKB approximation becomes better for large l and n , the convergence is basically fast. In the present approximation truncated at the next-to-leading order in the WKB expansion, the remaining terms cause the error in the estimation of $\langle \phi^2 \rangle$ of order $O(l_{\max}^{-2}, n_{\max}^{-2})$. As we increase the order of WKB approximation, the error decreases more rapidly. However, the price to pay for a faster numerical convergence is that the computation of the analytic part becomes more complicated. For small x , the above argument is completely correct. For larger x , however, $\Phi(l)$ is dominated not by ϖ but by σ in wide range of l and n . When σ is dominant, WKB expansion is not a good approximation at all. (WKB series does not converge in this case.) Therefore even if we use higher order WKB approximant, we need to sum the difference up to very large l and n .

In this case, in order to reduce the computational cost, we can use the interpolation of the values of difference evaluated only at sparse grid points of l and n . $\mathcal{P}_{3,2}$ also has a summation over n but again it is composed of the difference between the expression (36) and its truncation at $j = j_{\max}$. As in the case of Δ_1 , as we increase j_{\max} , the convergence of the summation over n becomes faster. Due to this property of fast convergence, no special trick is necessary in numerical evaluation of this term.

The latter part should be treated fully analytically due to the presence of infinite summation. However, it is not so difficult because they are all already written in terms of generalized zeta-functions, as we have seen in the previous subsection. These generalized zeta-functions, \mathcal{Z}_q , are functions of

$$v^2 \equiv \frac{x^2 \sigma}{\alpha^2}.$$

We need to evaluate them as a function of v^2 only once at the beginning of the whole calculation. Hence, the direct sum is one possible way to compute them. However, there are slightly cleverer ways to reduce the computational cost, which we will explain below.

One way to truncate this summation is as follows. Formally we write the summation defining $\tilde{\mathcal{Z}}_q$ as $\sum_{n=1}^{\infty} F_q(n)$. This summation can be directly performed numerically up to a certain value n_* . For the remaining part of summation, we simply expand $F_q(n)$ in terms of v as

$$F_q(n) = (\alpha v)^{-q} \sum_{k=1} d_{qk} \left(\frac{v}{n}\right)^{2k+1}, \quad (41)$$

where the coefficients d_{qk} are independent of α, v and n . Then, we have

$$\begin{aligned} \sum_{n=1}^{\infty} F_q(n) &\approx \sum_{n=1}^{n_*} F_q(n) \\ &+ \alpha^{-q} \sum_{k=1}^{k_{\max}} d_{qk} \tilde{\zeta}(2k+1) v^{2k-q+1}, \end{aligned} \quad (42)$$

where

$$\tilde{\zeta}(s) \equiv \sum_{n=n_{\max}+1}^{\infty} \frac{1}{n^s} = \zeta(s) - \sum_{n=1}^{n_{\max}} \frac{1}{n^s}.$$

In practice, we need to truncate the summation over k at a certain value k_{\max} , which is not necessarily so large as long as n_* is sufficiently large. In order that the expansion (41) converges for $\forall n \geq n_*$, n_*^2 must be larger than $|v^2|$. Hence, this method works efficiently unless $|v^2|$ is very large.

On the other hand, for a very large positive value of v^2 , we can make use of a simplified version of the Chowla-Selberg formula (see for example [40, 42]),

$$S(s, \rho) \equiv \sum_{n=1}^{\infty} (n^2 + \rho^2)^{-s}$$

$$\begin{aligned} &= -\frac{\rho^{-2s}}{2} + \frac{\sqrt{\pi}}{2} \frac{\Gamma(s-1/2)}{\Gamma(s)} \rho^{1-2s} \\ &+ \frac{2\pi^s}{\Gamma(s)} \rho^{-s+1/2} \sum_{p=1}^{\infty} p^{s-1/2} K_{s-1/2}(2\pi p \rho). \end{aligned} \quad (43)$$

The above identity basically rearranges the original sum in a convenient way, as the summation on the R.H.S. of the above equation converges very rapidly due to the exponential fall-off of the modified Bessel functions unless ρ is extremely small.

Among $\tilde{\mathcal{Z}}_q$, $\tilde{\mathcal{Z}}_{-1}$ requires the most delicate treatment. By definition,

$$\tilde{\mathcal{Z}}_{-1} = \alpha \sum_{n=1}^{\infty} \left(\sqrt{n^2 + v^2} - n - \frac{v^2}{2n} \right). \quad (44)$$

The Chowla-Selberg formula does not directly apply to this expression since the R.H.S. of (43) diverges for $s = 1/2$. Of course, it is due to this divergence why we subtracted counter terms. The trick is to regularize the above expression as

$$\begin{aligned} \tilde{\mathcal{Z}}_{-1}^{\text{reg}} &= \alpha \sum_{n=1}^{\infty} \left((n^2 + v^2)^{\frac{1}{2}-\epsilon} - n^{1-2\epsilon} - \left(\frac{1}{2} - \epsilon \right) v^2 n^{-1-2\epsilon} \right) \\ &= S(-1/2 + \epsilon, v^2) - \zeta_R(-1 + 2\epsilon) \\ &\quad - \left(\frac{1}{2} - \epsilon \right) v^2 \zeta_R(1 + 2\epsilon). \end{aligned}$$

This expression reduces to $\tilde{\mathcal{Z}}_{-1}$ if we set ϵ to 0, and it is finite for any non-negative ϵ . Hence, the limit $\epsilon \rightarrow 0$ gives $\tilde{\mathcal{Z}}_{-1}$. We can use the Chowla-Selberg formula, take the limit, and obtain an explicitly regular expression,

$$\begin{aligned} \tilde{\mathcal{Z}}_{-1} &= \alpha \left(\frac{1}{12} - \frac{v}{2} + v^2 \frac{1 - 2(\ln(v/2) + \gamma_E)}{4} \right. \\ &\quad \left. - \frac{v}{\pi} \sum_{p=1}^{\infty} \frac{K_{-1}(2\pi p v)}{p} \right). \end{aligned} \quad (45)$$

Rearranging the generalized zeta functions by means of the Chowla-Selberg formula, *i.e.* as a series of modified Bessel functions, proves to be efficient when the argument of the modified Bessel functions is real and not small.

For $\tilde{\mathcal{Z}}_1$, the regularized expression is simply given by

$$\tilde{\mathcal{Z}}_1^{\text{reg}} = S(1/2 + \epsilon, v) - \zeta_R(1 + 2\epsilon).$$

Taking the limit $\epsilon \rightarrow 0$, we obtain

$$\tilde{\mathcal{Z}}_1 = -\alpha^{-1} \left(\frac{1}{2v} + \ln(v/2) + \gamma_E - 2 \sum_{p=1}^{\infty} K_0(2\pi p v) \right) \quad (46)$$

The other generalized zeta functions \mathcal{Z}_q with $q > 1$, which do not require further regularization, are evaluated directly applying the Chowla-Selberg formula. We find

$$\mathcal{Z}_q = \alpha^{-q} \left(-\frac{v^{-q}}{2} + \frac{\sqrt{\pi} \Gamma((q-1)/2)}{2\Gamma(q/2)} v^{1-q} \right)$$

$$+ \frac{2\pi^{q/2}}{\Gamma(q/2)} v^{\frac{1-q}{2}} \sum_{p=1}^{\infty} p^{\frac{q-1}{2}} K_{\frac{q-1}{2}}(2\pi p v) \Big) . \quad (47)$$

Finally, we mention how to evaluate Δ_1 . It can be evaluated in an analogous manner. The first and the second terms in Δ_1 have exactly the same forms as \tilde{Z}_{-1} and \tilde{Z}_1 , apart from the replacement of v^2 with $\hat{m}^2 f / \alpha^2$. We write it here more explicitly as

$$\Delta_1 = \left[-\frac{2\alpha}{f} \tilde{Z}_{-1} + \left(\xi - \frac{1}{6} \right) \hat{R} \tilde{Z}_1 \right]_{v^2 \rightarrow \hat{m}^2 f / \alpha^2} . \quad (48)$$

VI. RESULTS

So far, we have not specified the form of the metric. For the numerical evaluation, however, we need to fix the function $f(r)$. To illustrate the results we will choose the case of a Sch-AdS black hole, with metric given by (1) and f by (2). For notational convenience we define the quantity $\ell_{bh} \equiv 2Mm_p^{-2}k$, which characterizes the size of the black hole.

The numerical analysis does not present particular difficulties and can be performed in a straightforward manner. The WKB part is expressed in terms of analytic functions and can thus be evaluated very easily. Slightly more involved is the computation of the sums. The WKB approximation to next-to-leading order ensures that the convergence goes relatively fast, as $O(l^{-5}, n^{-5})$. Therefore, after obtaining the solutions for the modes numerically, the summations can be computed directly.

After performing standard numerical checks, the various terms contributing to $\langle \phi^2 \rangle$ can be combined and some illustrative curves are reported for some representative values in the case of conformally coupled scalars ($\xi = 1/6$ and $m = 0$) in Fig. 1. (For $n = l = 0$ and very small values of M , the function Φ may become negative. In this case, the $n = l = 0$ mode has to be treated separately in the single sum Υ_0).

Due to the complexity of the calculation, it is instructive to look at the behavior of the vacuum polarization far away from the black hole, where we can expect to reproduce the leading AdS space result, which can be calculated independently. In fact, in pure AdS space the Green function is given by

$$G(X, X') = \frac{2}{3\pi^2} \left[3 - \cosh \left(k \sqrt{d(X, X')} \right) \right]^{-1} ,$$

where $d(X, X')$ is the geodesic distance between X and X' . By subtracting, as in the black hole case, the counterterms and then taking the coincidence limit, one gets:

$$\langle \phi^2 \rangle_{AdS} \simeq -\frac{1}{48\pi^2} . \quad (49)$$

Our results for Sch-AdS case can be fitted in the asymptotic limit as

$$\langle \phi^2 \rangle \simeq C_1 + C_2/f . \quad (50)$$

with $C_1 = -\frac{1}{48\pi^2}$. The first term represents the leading contribution to the vacuum polarization. We have checked the universality of C_1 to high accuracy.

The coefficient C_2 of the second term is hard to anticipate before calculation. If we use the result for the *vacuum polarization* at a finite temperature in Minkowski space, setting the temperature to the local Hawking temperature $k\alpha/2\pi\sqrt{f}$, we obtain the estimate $C_2 = \frac{\alpha^2}{48}$. However, our numerical result for C_2 is much smaller than this value in general. Since the temperature at large distance from the black hole is very low, a typical energy scale for the excitation is much below the inverse curvature length. Thus, it is not surprising that the rough estimate based on the result in Minkowski space does not hold in the present case.

VII. DISCUSSIONS AND CONCLUSIONS

In this paper we developed a method to compute the renormalized expectation value of a quantum scalar field, with mass m and coupling to the curvature ξ , in a thermal state on a spherically symmetric, asymptotically AdS black hole geometry.

We followed the approach of Refs. [3, 13], and employed the analytic WKB approximation and the point-splitting regularization to construct a regular expression for the coincidence limit of the Green function. We explicitly demonstrated the regularity of the Green function, and this allowed us to perform the renormalization term by term.

Analogously to the asymptotically flat case, the WKB approximation arranges the vacuum polarization in a ‘WKB-part’ plus a remainder. One term depends on the WKB approximants and it can be evaluated analytically, although the computations become more cumbersome as the order of the approximation increases.

It is very interesting to notice that the WKB approximation organizes the analytical part of the vacuum polarization as a series of analytic functions. These functions take the form of generalized zeta functions, that occur in the computation of functional determinants of Laplacians on Riemann spheres. The coefficients of the expansions can be calculated order by order in the WKB expansion. We explicitly showed this to next-to-leading order in the WKB approximations. The other term is a remainder of the WKB approximation, in the sense that it depends on the difference between the exact solutions for the modes and their WKB counterpart.

In the end, both terms have to be handled numerically. The generalized zeta functions can be easily evaluated using direct summation when their argument is small. When the argument is not small, it is convenient to rearrange these functions by means of a simplified version of the Chowla-Selberg formula [40]. The rearranged expression contains an infinite series of modified Bessel functions, but it converges exponentially fast when the argument is not small. The ‘remainder’ can be evalu-

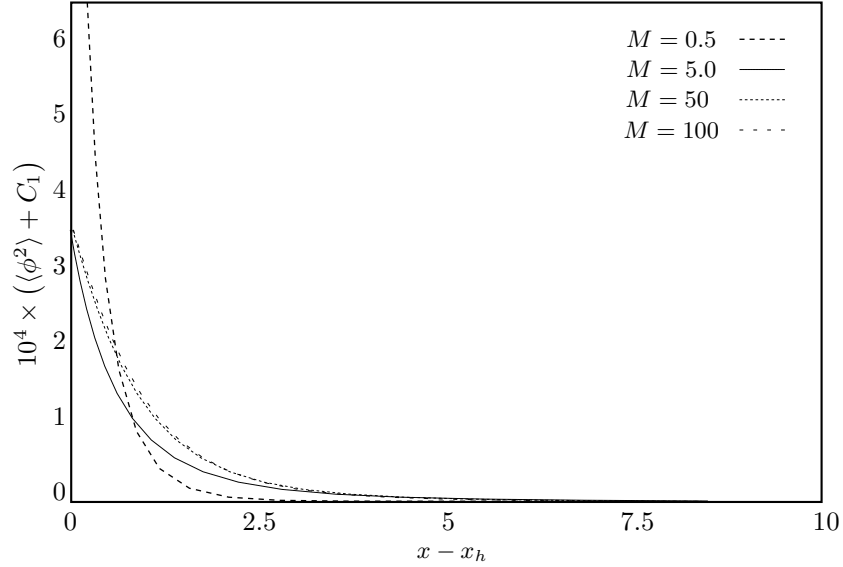


FIG. 1: The figure illustrates the behavior of $\langle\phi^2\rangle$ for conformally coupled ($\xi = 1/6$ and $m = 0$) fields. The curves refer to: $M = 100, 50, 5, 0.5$.

ated by directly solving the mode equation numerically and taking the difference with the WKB counterparts. These terms converge very rapidly, as we expected from the fact that the WKB approximation works very well: at next-to-leading order the remainder is $O(l^{-5}, n^{-5})$.

In the last part, we explicitly illustrated the method by computing the vacuum polarization for conformal fields on a Sch-AdS black hole background. We finally discussed the asymptotic behavior of the vacuum polarization and compared it with the pure AdS result which can be calculated independently. We find that our result reproduces the leading universal (constant) behavior to high accuracy.

We are now using the methodology presented in this paper to compute the energy momentum tensor for a quantum field on an asymptotically AdS black hole ge-

ometry and we hope to report on this soon.

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